Available online at www.sciencedirect.com

ScienceDirect

Linear Algebra and its Applications 424 (2007) 320–337

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Idempotency of linear combinations of three idempotent matrices, two of which are commuting

Oskar Maria Baksalary [a,*], Julio Benítez [b]

[a] *Institute of Physics, Adam Mickiewicz University, ul. Umultowska 85, PL 61-614 Poznań, Poland*

[b] *Departamento de Matemática Aplicada, Instituto de Matemática Multidisciplinar, Universidad Politécnica de Valencia, Camino de Vera s/n. 46022, Valencia, Spain*

Received 18 July 2006; accepted 9 February 2007

Available online 21 February 2007

Submitted by R.A. Brualdi

Dedicated to Professor Roger Horn on the occasion of his 65th birthday

Abstract

The considerations of the present paper were inspired by Baksalary [O.M. Baksalary, Idempotency of linear combinations of three idempotent matrices two of which are disjoint, Linear Algebra Appl. 388 (2004) 67–78] who characterized all situations in which a linear combination $\mathbf{P} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2 + c_3\mathbf{P}_3$, with c_i, $i = 1, 2, 3$, being nonzero complex scalars and $\mathbf{P}_i$, $i = 1, 2, 3$, being nonzero complex idempotent matrices such that two of them, $\mathbf{P}_1$ and $\mathbf{P}_2$ say, are disjoint, i.e., satisfy condition $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0} = \mathbf{P}_2\mathbf{P}_1$, is an idempotent matrix. In the present paper, by utilizing different formalism than the one used by Baksalary, the results given in the above mentioned paper are generalized by weakening the assumption expressing the disjointness of $\mathbf{P}_1$ and $\mathbf{P}_2$ to the commutativity condition $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$.

AMS classification: 15A57; 15A27

Keywords: Oblique projector; Orthogonal projector; Partitioned matrix

* Corresponding author. Tel.: +48 61 829 50 76; fax: +48 61 829 50 70.

E-mail addresses: baxx@amu.edu.pl (O.M. Baksalary), jbenitez@mat.upv.es (J. Benítez).

0024-3795/$ - see front matter © 2007 Elsevier Inc. All rights reserved.

doi:10.1016/j.laa.2007.02.016

1. Introduction

Let $\mathbb{C}_{m,n}$ denote the set of $m \times n$ complex matrices. The symbol $\mathbb{C}_n^{\mathsf{P}}$ will stand for the subset of $\mathbb{C}_{n,n}$ consisting of idempotent matrices (oblique projectors), i.e.,

$$\mathbb{C}_n^{\mathsf{P}} = \{\mathbf{P} \in \mathbb{C}_{n,n}\colon\ \mathbf{P}^2 = \mathbf{P}\},$$

whereas $\mathbb{C}_n^{\mathsf{OP}}$ for the subset of $\mathbb{C}_n^{\mathsf{P}}$ composed of Hermitian idempotent matrices (orthogonal projectors), i.e.,

$$\mathbb{C}_n^{\mathsf{OP}} = \{\mathbf{P} \in \mathbb{C}_{n,n}\colon\ \mathbf{P}^2 = \mathbf{P} = \mathbf{P}^*\},$$

where $\mathbf{P}^*$ is the conjugate transpose of $\mathbf{P}$. Moreover, $\mathbf{I}_n$ will mean the identity matrix of order n and $\mathrm{r}(\mathbf{K})$ will be the rank of $\mathbf{K} \in \mathbb{C}_{m,n}$.

The considerations of this paper were inspired by Baksalary [2] who considered the problem of characterizing all situations in which a linear combination of the form

$$\mathbf{P} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2 + c_3\mathbf{P}_3, \tag{1.1}$$

with nonzero $c_i \in \mathbb{C}$, $i = 1, 2, 3$, and nonzero $\mathbf{P}_i \in \mathbb{C}_n^{\mathsf{P}}$, $i = 1, 2, 3$, such that two of them, $\mathbf{P}_1$ and $\mathbf{P}_2$ say, are disjoint, i.e., satisfy condition

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{0} = \mathbf{P}_2\mathbf{P}_1, \tag{1.2}$$

is an idempotent matrix. In the present paper, the results given in [2] are generalized by establishing the complete solution to the problem of when a linear combination of the form (1.1) satisfies $\mathbf{P}^2 = \mathbf{P}$ with the assumption (1.2) replaced by an essentially weaker commutativity condition

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1. \tag{1.3}$$

The generalization was obtained by utilizing different formalism than the one used in [2], in which projectors $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ are represented as partitioned matrices. Such a representation turns out to be very powerful in dealing with problems similar to the one under consideration.

It is noteworthy that the problem of characterizing situations in which a linear combination of the form (1.1) is an idempotent matrix was independently considered by Özdemir and Özban [4], with the use of yet another formalism. However, their approach has limited applicability, for it can be utilized exclusively to the situations in which matrices $\mathbf{P}_i$, $i = 1, 2, 3$, occurring in (1.1), are different, mutually commuting, i.e., satisfy

$$\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_j\mathbf{P}_i, \quad i \neq j, \ \ i, j = 1, 2, 3, \tag{1.4}$$

and such that either

$$\mathbf{P}_i\mathbf{P}_j = \mathbf{0} \quad \text{or} \quad \mathbf{P}_i\mathbf{P}_j = \mathbf{P}_i, \quad i \neq j, \ \ i, j = 1, 2, 3.$$

Furthermore, due to the intrinsic limitations of the formalism, the authors were able to characterize only some particular sets of sufficient conditions ensuring that $\mathbf{P}^2 = \mathbf{P}$; see Theorem 3.2 in [4].

It should be emphasized that an essential motivation to generalize the problem posed in [2] originates from statistics, where considerations concerning the inheritance of the idempotency by linear combinations of idempotent matrices have very useful applications in the theory of distributions of quadratic forms in normal variables; see e.g., Lemma 9.1.2 in [5] or p. 68 in [2]. Thus, it is of interest to explore the problem under consideration as extensively as possible.

In the next section we provide three theorems constituting the main result of the paper and show that the extent in which they generalize Theorem 1 in [2] is significant. Section 3 contains some additions results referring to the situations in which matrices $\mathbf{P}_i$, $i = 1, 2, 3$, occurring in (1.1), belong to the set $\mathbb{C}_n^{\mathsf{OP}}$, being of particular interest from the point of view of possible applications in statistics.

2. Main result

The main result of Baksalary [2] is given therein as Theorem 1, which is split into four disjoint parts (a)–(d) referring to situations in which matrices $\mathbf{P}_i$, $i = 1, 2, 3$, occurring in (1.1), in addition to (1.2), satisfy also conditions:

(a) $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3\mathbf{P}_1$, $\mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_3\mathbf{P}_2$,
(b) $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3\mathbf{P}_1$, $\mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2$,
(c) $\mathbf{P}_1\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_1$, $\mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_3\mathbf{P}_2$,
(d) $\mathbf{P}_1\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_1$, $\mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2$.

The complete solution to the problem considered in this paper is given in three subsequent theorems, of which Theorem 1 correspond to part (a) of Theorem 1 in [2], Theorem 2 to parts (b) and (c), and Theorem 3 to part (d). As already mentioned, Theorems 1–3 generalize Theorem 1 in [2] and the generalization is included in replacing condition (1.2) by an essentially weaker condition (1.3).

A theorem below generalizes part (a) of Theorem 1 in [2] and Theorem 3.2 in [4].

Theorem 1. *Let* $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{P}}$ *be nonzero and mutually commuting, i.e., satisfying conditions* (1.4). *Moreover, let* $\mathbf{P}$ *be a linear combination of the form* (1.1), *with nonzero* $c_1, c_2, c_3 \in \mathbb{C}$. *Then the following list comprises characteristics of all cases in which* $\mathbf{P}$ *is an idempotent matrix*:

(a) $\mathbf{P}_i + \mathbf{P}_j\mathbf{P}_k = \mathbf{P}_i\mathbf{P}_j + \mathbf{P}_i\mathbf{P}_k$ *holds along with* $c_i = -1$, $c_j = 1$, $c_k = 1$,
(b) $\mathbf{P}_i = \mathbf{P}_i\mathbf{P}_k + \mathbf{P}_j$, $\mathbf{P}_j\mathbf{P}_k = \mathbf{0}$, *hold along with* $c_i = -1$, $c_j = 2$, $c_k = 1$,
(c) $\mathbf{P}_i + 2\mathbf{P}_j\mathbf{P}_k = \mathbf{P}_j + \mathbf{P}_k$, $c_j = \frac{1}{2}$, $c_k = \frac{1}{2}$, *hold along with* $c_i = -\frac{1}{2}$ *or* $c_i = \frac{1}{2}$,
(d) $\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_j$, $\mathbf{P}_i\mathbf{P}_k = \mathbf{P}_k$, $\mathbf{P}_j\mathbf{P}_k = \mathbf{0}$, *hold along with* $c_i = 1$, $c_j = -1$, $c_k = -1$,
(e) $\mathbf{P}_i\mathbf{P}_k = \mathbf{P}_j$ *holds along with* $c_i = 1$, $c_j = -2$, $c_k = 1$,
(f) $\mathbf{P}_i + \mathbf{P}_j\mathbf{P}_k = \mathbf{P}_j + \mathbf{P}_k$ *holds along with* $c_i = 2$, $c_j = -1$, $c_k = -1$,
(g) $\mathbf{P}_j = \mathbf{P}_k$, $\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_j$, *hold along with* $c_j + c_k = -1$, $c_i = 1$,
(h) $\mathbf{P}_i = \mathbf{P}_k$ *holds along with* $c_i + c_k = 0$, $c_j = 1$,
(i) $\mathbf{P}_i = \mathbf{P}_k$, $\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_j$, *hold along with* $c_i + c_k = 1$, $c_j = -1$,
(j) $\mathbf{P}_j = \mathbf{P}_k$, $\mathbf{P}_i\mathbf{P}_j = \mathbf{0}$, *hold along with* $c_j + c_k = 1$, $c_i = 1$,
(k) $\mathbf{P}_i = \mathbf{P}_j + \mathbf{P}_k$, $\mathbf{P}_j\mathbf{P}_k = \mathbf{0}$, *hold along with* $c_i + c_j = 0$, $c_i + c_k = 0$ *or* $c_i + c_j = 0$, $c_i + c_k = 1$ *or* $c_i + c_j = 1$, $c_i + c_k = 1$,
(l) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{0}$, $\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$, *hold along with* $c_1 = 1$, $c_2 = 1$, $c_3 = 1$,
(m) $\mathbf{P}_1 = \mathbf{P}_2 = \mathbf{P}_3$ *hold along with* $c_1 + c_2 + c_3 \in \{0, 1\}$,

where in characteristics (a)–(k) $i \neq j$, $i \neq k$, $j \neq k$, $i, j, k = 1, 2, 3$.

Proof. Straightforward calculations show that matrix $\mathbf{P}$ of the form (1.1) is idempotent if and only if

$$c_1(c_1 - 1)\mathbf{P}_1 + c_2(c_2 - 1)\mathbf{P}_2 + c_3(c_3 - 1)\mathbf{P}_3 + c_1c_2(\mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_2\mathbf{P}_1) + c_1c_3(\mathbf{P}_1\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_1) + c_2c_3(\mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2) = \mathbf{0}. \tag{2.1}$$

Clearly, taking into account assumptions (1.4), Eq. (2.1) reduces to

$$\begin{aligned} &c_1(c_1-1)\mathbf{P}_1 + c_2(c_2-1)\mathbf{P}_2 + c_3(c_3-1)\mathbf{P}_3 \\ &\quad + 2c_1c_2\mathbf{P}_1\mathbf{P}_2 + 2c_1c_3\mathbf{P}_1\mathbf{P}_3 + 2c_2c_3\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}. \end{aligned} \tag{2.2}$$

Sufficiency of the conditions revealed in 13 characteristics provided in the theorem follows by direct verification of criterion (2.2). In the proof of necessity, we utilize the fact that every idempotent matrix is diagonalizable (see e.g., [6, Theorem 4.1]), and thus there exists a nonsingular matrix $\mathbf{W} \in \mathbb{C}_{n,n}$ such that

$$\mathbf{P}_1 = \mathbf{W}(\mathbf{I}_r \oplus \mathbf{0})\mathbf{W}^{-1}, \tag{2.3}$$

where $r = \mathrm{r}(\mathbf{P}_1)$ and "$\oplus$" denotes a direct sum. Clearly, $0 < r \leqslant n$ and if $r = n$, then the latter of the summands in representation (2.3) vanishes. Since $\mathbf{P}_1$, $\mathbf{P}_2$ as well as $\mathbf{P}_1$, $\mathbf{P}_3$ commute, we can represent $\mathbf{P}_2$ and $\mathbf{P}_3$ as

$$\mathbf{P}_2 = \mathbf{W}(\mathbf{X} \oplus \mathbf{Y})\mathbf{W}^{-1} \quad \text{and} \quad \mathbf{P}_3 = \mathbf{W}(\mathbf{S} \oplus \mathbf{T})\mathbf{W}^{-1}, \tag{2.4}$$

with $\mathbf{X}, \mathbf{S} \in \mathbb{C}_{r,r}$, $\mathbf{Y}, \mathbf{T} \in \mathbb{C}_{n-r,n-r}$, where $\mathbf{Y}$ and $\mathbf{T}$ vanish when $\mathbf{P}_1$ in (2.3) is nonsingular. From the idempotency of matrices $\mathbf{P}_2$ and $\mathbf{P}_3$ it follows that $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{S}$, and $\mathbf{T}$ are all idempotent, whereas conditions (1.4) ensure that $\mathbf{XS} = \mathbf{SX}$ and $\mathbf{YT} = \mathbf{TY}$. An additional useful observation concerning matrices given in (2.3) and (2.4) is that the idempotency of a linear combination (1.1) is equivalent to the conjunction $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S} \in \mathbb{C}_r^{\mathsf{P}}$ and $c_2\mathbf{Y} + c_3\mathbf{T} \in \mathbb{C}_{n-r}^{\mathsf{P}}$. In the consecutive steps of the proof we establish necessary and sufficient conditions ensuring that this conjunction is satisfied, expressed in terms of scalars c_1, c_2, c_3 and matrices $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{S}$, and $\mathbf{T}$, and then reexpress these conditions in terms of scalars c_1, c_2, c_3 and matrices $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. The final step of the proof, based on the observation that (2.2) is invariant with respect to the interchange of indexes "1" and "2", "1" and "3", "2" and "3", consists in replacing indexes "1", "2", and "3" by "i", "j", and "k", respectively, where $i \neq j$, $i \neq k$, $j \neq k$, $i, j, k = 1, 2, 3$.

In the first step of the proof observe that, in view of $\mathbf{YT} = \mathbf{TY}$, matrix $c_2\mathbf{Y} + c_3\mathbf{T}$ is idempotent if and only if any of the following sets of conditions holds:

$$\mathbf{Y} = \mathbf{0},\ \mathbf{T} = \mathbf{0},\ c_2, c_3 \in \mathbb{C} \setminus \{0\}, \tag{2.5}$$

$$\mathbf{Y} = \mathbf{0},\ c_2 \in \mathbb{C} \setminus \{0\},\ c_3 = 1, \tag{2.6}$$

$$\mathbf{T} = \mathbf{0},\ c_2 = 1,\ c_3 \in \mathbb{C} \setminus \{0\}, \tag{2.7}$$

$$\mathbf{Y} = \mathbf{T},\ c_2 + c_3 \in \{0, 1\}, \tag{2.8}$$

$$\mathbf{YT} = \mathbf{0},\ c_2 = 1,\ c_3 = 1, \tag{2.9}$$

$$\mathbf{YT} = \mathbf{T},\ c_2 = 1,\ c_3 = -1, \tag{2.10}$$

$$\mathbf{YT} = \mathbf{Y},\ c_2 = -1,\ c_3 = 1. \tag{2.11}$$

Sets (2.5)–(2.8) constitute characterizations of situations in which a scalar multiple of an idempotent matrix is also idempotent, whereas sets (2.9)–(2.11) follow straightforwardly from Theorem in [1]. Clearly, conditions (2.6), (2.7) and (2.10), (2.11) are counterparts of each other obtained by interchanging simultaneously matrices $\mathbf{Y}$ and $\mathbf{T}$ as well as scalars c_2 and c_3. In consequence, only five sets from among (2.5)–(2.11) will be taken into account in further considerations.

In the second step note that the idempotency of $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$ can be equivalently expressed as

$$c_1(c_1-1)\mathbf{I}_r + c_2(2c_1+c_2-1)\mathbf{X} + c_3(2c_1+c_3-1)\mathbf{S} + c_2c_3(\mathbf{XS}+\mathbf{SX}) = \mathbf{0}, \tag{2.12}$$

from where, taking into account that $\mathbf{XS} = \mathbf{SX}$, we get

$$c_1(c_1 - 1)\mathbf{I}_r + c_2(2c_1 + c_2 - 1)\mathbf{X} + c_3(2c_1 + c_3 - 1)\mathbf{S} + 2c_2c_3\mathbf{XS} = \mathbf{0}. \tag{2.13}$$

Since $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$ commutes with both $\mathbf{X}$ and $\mathbf{S}$, multiplying this linear combination by $\mathbf{X}$ and $\mathbf{S}$ leads to

$$(c_1 + c_2)\mathbf{X} + c_3\mathbf{XS} \in \mathbb{C}_r^{\mathsf{P}} \quad \text{and} \quad (c_1 + c_3)\mathbf{S} + c_2\mathbf{XS} \in \mathbb{C}_r^{\mathsf{P}}, \tag{2.14}$$

respectively. Observe that the two conditions in (2.14) are symmetrical in the sense that one of them is obtainable from the other by simultaneous interchange of matrices $\mathbf{X}$ and $\mathbf{S}$ as well as scalars c_2 and c_3. Thus, we can restrict the considerations to, say, the condition on the left-hand side only. It is seen that this condition is satisfied if and only if any of the following sets of conditions holds:

$$\mathbf{X} = \mathbf{0},\ c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}, \tag{2.15}$$
$$\mathbf{XS} = \mathbf{0},\ c_1 + c_2 \in \{0, 1\},\ c_3 \in \mathbb{C} \setminus \{0\}, \tag{2.16}$$
$$\mathbf{XS} = \mathbf{X},\ c_1 + c_2 + c_3 \in \{0, 1\}, \tag{2.17}$$
$$c_1 + c_2 = 0,\ c_3 = 1, \tag{2.18}$$
$$c_1 + c_2 = 1,\ c_3 = -1, \tag{2.19}$$

where sets (2.15)–(2.18) constitute characterizations of situations in which a scalar multiple of an idempotent matrix is also idempotent, whereas set (2.19) follows from Theorem in [1].

If the first condition in (2.15) is satisfied, i.e., $\mathbf{X} = \mathbf{0}$, then the idempotency of $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$ is equivalent to the idempotency of $c_1\mathbf{I}_r + c_3\mathbf{S}$. It is easily seen that $c_1\mathbf{I}_r + c_3\mathbf{S} \in \mathbb{C}_r^{\mathsf{P}}$ is satisfied if and only if any of the following sets of conditions holds:

$$\mathbf{S} = \mathbf{0},\ c_1 = 1,\ c_3 \in \mathbb{C} \setminus \{0\}, \tag{2.20}$$
$$\mathbf{S} = \mathbf{I}_r,\ c_1 + c_3 \in \{0, 1\}, \tag{2.21}$$
$$c_1 = 1,\ c_3 = -1, \tag{2.22}$$

where sets (2.20) and (2.21) constitute characterizations of situations in which a scalar multiple of an idempotent matrix is also idempotent, and set (2.22) follows from Theorem in [1]. By combining sets (2.20)–(2.22) with (2.15) we obtain three sets of necessary and sufficient conditions for $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S} \in \mathbb{C}_r^{\mathsf{P}}$, being of the forms

$$\mathbf{X} = \mathbf{0},\ \mathbf{S} = \mathbf{0},\ c_1 = 1,\ c_2, c_3 \in \mathbb{C} \setminus \{0\}, \tag{2.23}$$
$$\mathbf{X} = \mathbf{0},\ \mathbf{S} = \mathbf{I}_r,\ c_1 + c_3 \in \{0, 1\},\ c_2 \in \mathbb{C} \setminus \{0\}, \tag{2.24}$$
$$\mathbf{X} = \mathbf{0},\ c_1 = 1,\ c_2 \in \mathbb{C} \setminus \{0\},\ c_3 = -1. \tag{2.25}$$

Now we consider set (2.16). In this situation, regardless whether $c_1 + c_2 = 0$ or $c_1 + c_2 = 1$, condition (2.13) reduces to

$$c_1(c_1 - 1)(\mathbf{I}_r - \mathbf{X}) + c_3(2c_1 + c_3 - 1)\mathbf{S} = \mathbf{0}, \tag{2.26}$$

and multiplying this equation by $\mathbf{S}$ also leads to a conclusion that $c_1 + c_3 = 0$ or $c_1 + c_3 = 1$ or $\mathbf{S} = \mathbf{0}$. Substituting, the first two of these relationships to (2.26) implies that either $c_1 = 1$ or $\mathbf{X} + \mathbf{S} = \mathbf{I}_r$. Observing that $c_1 = 1$ is in contradictions with $c_1 + c_3 = 1$ and $c_1 + c_2 = 1$, we arrive at another five characterizations of $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S} \in \mathbb{C}_r^{\mathsf{P}}$, being of the forms

$$\mathbf{XS} = \mathbf{0},\ c_1 = 1,\ c_2 = -1,\ c_3 = -1, \tag{2.27}$$

and

$$\mathbf{X} + \mathbf{S} = \mathbf{I}_r, \; c_1 + c_2 \in \{0, 1\}, \; c_1 + c_3 \in \{0, 1\}. \tag{2.28}$$

Substituting $\mathbf{S} = \mathbf{0}$ to (2.26) leads to an alternative $c_1 = 1$ or $\mathbf{X} = \mathbf{I}_r$, from where we obtain another three sets of conditions. However, these sets are not new, for they are counterparts of sets (2.25) and (2.24) obtained by interchanging $\mathbf{X}$ and $\mathbf{S}$ as well as c_2 and c_3.

Next we consider set (2.17). If $c_1 + c_2 + c_3 = 0$, then (2.13) takes the form

$$c_1(c_1 - 1)\mathbf{I}_r - c_2(c_2 + 1)\mathbf{X} - (c_1 + c_2)(c_1 - c_2 - 1)\mathbf{S} = \mathbf{0}, \tag{2.29}$$

whereas if $c_1 + c_2 + c_3 = 1$, then it reduces to

$$c_1(c_1 - 1)\mathbf{I}_r - c_2(c_2 - 1)\mathbf{X} - (c_1 - c_2)(c_1 + c_2 - 1)\mathbf{S} = \mathbf{0}. \tag{2.30}$$

Multiplying (2.29) and (2.30) by $\mathbf{S}$ leads to alternatives: $c_2 = -1$ or $\mathbf{X} = \mathbf{S}$ and $c_2 = 1$ or $\mathbf{X} = \mathbf{S}$, respectively. If $c_2 = -1$, then (2.29) entails $c_1 = 1$ or $\mathbf{S} = \mathbf{I}_r$, and similar alternative is obtained from (2.30) if $c_2 = 1$. Since conjunction $c_1 = 1, c_2 = -1$ is in contradiction with $c_1 + c_2 + c_3 = 0$, we have another three characterizations of the idempotency of $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$, namely

$$\mathbf{S} = \mathbf{I}_r \text{ and either } c_1 + c_3 = 1, \; c_2 = -1 \text{ or } c_1 + c_3 = 0, \; c_2 = 1, \tag{2.31}$$

$$\mathbf{XS} = \mathbf{X}, \; c_1 = 1, \; c_2 = 1, \; c_3 = -1. \tag{2.32}$$

If $\mathbf{X} = \mathbf{S}$, then (2.29) as well as (2.30) entail that either $c_1 = 1$ or $\mathbf{X} = \mathbf{I}_r$. Hence, we arrive at the following four characterizations:

$$\mathbf{X} = \mathbf{S}, \; c_1 = 1, \; c_2 + c_3 \in \{0, -1\}, \tag{2.33}$$

$$\mathbf{X} = \mathbf{I}_r, \; \mathbf{S} = \mathbf{I}_r, \; c_1 + c_2 + c_3 \in \{0, 1\}. \tag{2.34}$$

Two sets left to be considered, both characterized by scalars c_1, c_2, c_3 only. Substituting (2.18) to (2.13) gives

$$(c_1 - 1)\mathbf{I}_r - (c_1 - 1)\mathbf{X} + 2\mathbf{S} - 2\mathbf{XS} = \mathbf{0}, \tag{2.35}$$

and multiplying (2.35) by $\mathbf{S}$ entails an alternative $c_1 = -1$ or $\mathbf{XS} = \mathbf{S}$. In the former of these cases (2.35) yields $\mathbf{I}_r - \mathbf{X} - \mathbf{S} + \mathbf{XS} = \mathbf{0}$, and hence the next characterization of $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S} \in \mathbb{C}_r^{\mathsf{P}}$ is

$$(\mathbf{X} - \mathbf{I}_r)(\mathbf{S} - \mathbf{I}_r) = \mathbf{0}, \; c_1 = -1, \; c_2 = 1, \; c_3 = 1. \tag{2.36}$$

If now $\mathbf{XS} = \mathbf{S}$, then (2.35) implies that either $c_1 = 1$ or $\mathbf{X} = \mathbf{I}_r$. However, both these situations lead to sets already listed, namely the former of them to a counterpart of (2.32), whereas the latter to a counterpart of the right-hand side characterization in (2.31).

To complete this step of the proof we consider set (2.19). Since $c_1 \neq 1$, in this situation (2.13) reduces to

$$c_1\mathbf{I}_r - c_1\mathbf{X} - 2\mathbf{S} + 2\mathbf{XS} = \mathbf{0}, \tag{2.37}$$

and multiplying this equation by $\mathbf{S}$ entails an alternative $c_1 = 2$ or $\mathbf{XS} = \mathbf{S}$. In the former of these cases, (2.37) yields $(\mathbf{X} - \mathbf{I}_r)(\mathbf{S} - \mathbf{I}_r) = \mathbf{0}$, leading to the set of the form

$$(\mathbf{X} - \mathbf{I}_r)(\mathbf{S} - \mathbf{I}_r) = \mathbf{0}, \; c_1 = 2, \; c_2 = -1, \; c_3 = -1, \tag{2.38}$$

whereas in the latter case, (2.37) yields $\mathbf{X} = \mathbf{I}_r$, leading to set being a counterpart of the left-hand side characterization in (2.31).

The present step of the proof is concluded by a clear observation that the list of conditions necessary and sufficient for $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S} \in \mathbb{C}_r^{\mathsf{P}}$ given in (2.23)–(2.25), (2.27), (2.28), (2.31)–(2.34), (2.36), and (2.38) is not complete and the remaining characterizations are obtainable from the available ones by interchanging simultaneously matrices $\mathbf{X}$, $\mathbf{S}$ and scalars c_2, c_3. However, sets (2.23), (2.27), (2.28), (2.33), (2.34), (2.36), and (2.38) are invariant with respect to such interchanges and thus only six additional characterizations follow.

In the next step of the proof we establish necessary and sufficient conditions ensuring that $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$ and $c_2\mathbf{Y} + c_3\mathbf{T}$ are both idempotent. This aim is accomplished by combining each of the 11 characterizations (2.23)–(2.25), (2.27), (2.28), (2.31)–(2.34), (2.36), and (2.38) with each of the five sets (2.5), (2.6), and (2.8)–(2.10). First observe that combining (2.5) or (2.6) with any characterization from (2.23)–(2.25) leads to situations in which $\mathbf{X} = \mathbf{0}$, $\mathbf{Y} = \mathbf{0}$, being in a contradiction with the assumption that $\mathbf{P}_2$ is nonzero. Next observe that the pairs composed of: any characterization from (2.6), (2.8)–(2.10) and either (2.27) or (2.38); (2.6) and (2.32); (2.8) and (2.36); (2.9) and (2.25), (2.32), or (2.33); (2.10) and either (2.28) or (2.36) are irreconcilable. In consequence, 34 conjunctions of characterizations remain to be considered.

Combining (2.6) with a version of (2.28) having $c_1 + c_2 = 0$, yields $c_1 = -1, c_2 = 1, c_3 = 1$, $\mathbf{X} + \mathbf{S} = \mathbf{I}_r$, $\mathbf{Y} = \mathbf{0}$. Since $\mathbf{X} + \mathbf{S} = \mathbf{I}_r \Rightarrow \mathbf{XS} = \mathbf{0}$, it is clear that in this situation $\mathbf{I}_r + \mathbf{XS} = \mathbf{X} + \mathbf{S}$, $\mathbf{YT} = \mathbf{0}$ also hold. These two matrix equations can be equivalently expressed in terms of matrices $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ given in (2.3) and (2.4) as $\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3$. Hence, replacing indexes “1”, “2”, and “3” by “i”, “j”, and “k”, respectively, where $i \neq j$, $i \neq k$, $j \neq k$, $i, j, k = 1, 2, 3$, leads to characteristic (a) of the theorem. The same characteristic is obtained by combining: (2.5) with either (2.32) or (2.36); (2.6) with (2.31), a version of (2.33) having $c_2 + c_3 = 0$, or (2.36); (2.8) with (2.25) under $c_2 + c_3 = 0$, a version of (2.31) having $c_1 + c_3 = 0$, $c_2 = 1$, or (2.32); (2.9) with (2.24), (2.28), (2.31), a version of (2.34) having $c_1 + c_2 + c_3 = 1$, or (2.36); and (2.10) with (2.23), a version of (2.24) having $c_1 + c_3 = 0$, (2.25), or any characteristic from (2.31)–(2.34).

Combining again (2.6), but this time with a version of (2.28) having $c_1 + c_2 = 1$, entails $c_1 = -1, c_2 = 2, c_3 = 1, \mathbf{X} + \mathbf{S} = \mathbf{I}_r, \mathbf{Y} = \mathbf{0}$. It is easily seen that the matrix conditions are equivalent to the conjunction $\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_3 + \mathbf{P}_2$, $\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$. Hence, introducing indexes “i”, “j”, and “k”, we establish characteristic (b) of the theorem. The same characteristic follows by combining (2.8) with (2.25) under $c_2 + c_3 = 1$; (2.10) with a version of (2.24) having $c_1 + c_3 = 1$.

Next we consider combination of (2.8) with (2.28), which leads to $c_1 + c_2 \in \{0, 1\}, c_1 + c_3 \in \{0, 1\}, c_2 + c_3 \in \{0, 1\}, \mathbf{X} + \mathbf{S} = \mathbf{I}_r, \mathbf{Y} = \mathbf{T}$. The three conditions on scalars are satisfied merely in four situations, namely when c_1, c_2, c_3 are all equal to $\frac{1}{2}$ or when two scalars are equal to $\frac{1}{2}$ while the third one is equal to $-\frac{1}{2}$. On the other hand, $\mathbf{X} + \mathbf{S} = \mathbf{I}_r \Rightarrow \mathbf{X} + \mathbf{S} = \mathbf{I}_r + 2\mathbf{XS}$ and $\mathbf{Y} = \mathbf{T} \Rightarrow 2\mathbf{YT} = \mathbf{Y} + \mathbf{T}$, with the conjunction of the conditions on the right-hand sides of the implications being equivalent to $\mathbf{P}_1 + 2\mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_2 + \mathbf{P}_3$. Hence, we arrive and characteristic (c) of the theorem.

Combining now (2.5) with (2.27) entails $c_1 = 1$, $c_2 = -1$, $c_3 = -1$, $\mathbf{XS} = \mathbf{0}$, $\mathbf{Y} = \mathbf{0}$, $\mathbf{T} = \mathbf{0}$. Utilizing matrices given in (2.3) and (2.4), it follows that these matrix conditions are equivalent to $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2, \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3, \mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$. In consequence, the characteristic (d) is established.

Let us now combine (2.6) with a version of (2.33) having $c_2 + c_3 = -1$. In such situation $c_1 = 1$, $c_2 = -2$, $c_3 = 1$, $\mathbf{X} = \mathbf{S}$, $\mathbf{Y} = \mathbf{0}$, with the matrix conditions being equivalent to $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2$. Hence, consequently taking $i = 1$, $j = 2$, and $k = 3$, we obtain characteristic (e). The same characteristic follows by combining (2.9) with a version of (2.34) having $c_1 + c_2 + c_3 = 0$.

Yet another characteristic involving definite values of scalars c_1, c_2, c_3 is obtained by combining, for instance, (2.5) with (2.38). Then, $c_1 = 2, c_2 = -1, c_3 = -1$, $(\mathbf{X} - \mathbf{I}_r)(\mathbf{S} - \mathbf{I}_r) = \mathbf{0}, \mathbf{Y} = \mathbf{0}$, $\mathbf{T} = \mathbf{0}$. Since the last two matrix conditions trivially ensure that $\mathbf{YT} = \mathbf{Y} + \mathbf{T}$, in turn we get $\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_2 + \mathbf{P}_3$. Thus, characteristic (f) of the theorem is shown. The same characteristic is obtained as a result of combining (2.8) with (2.31) having $c_1 + c_3 = 1, c_2 = -1$.

Further, when (2.5) is combined with (2.33) having $c_2 + c_3 = -1$, then in addition to $c_2 + c_3 = -1$, we get $c_1 = 1$, $\mathbf{X} = \mathbf{S}$, $\mathbf{Y} = \mathbf{0}$, $\mathbf{T} = \mathbf{0}$, with these matrix conditions satisfied if and only if $\mathbf{P}_2 = \mathbf{P}_3$, $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$. Hence, characteristic (g) follows. The same characteristic is obtained by combining (2.6) with a version of (2.34) having $c_1 + c_2 + c_3 = 0$.

Next we combine (2.5) with a version of (2.31) having $c_1 + c_3 = 0, c_2 = 1$. Then these conditions are supplemented by $\mathbf{S} = \mathbf{I}_r$, $\mathbf{Y} = \mathbf{0}$, $\mathbf{T} = \mathbf{0}$. Clearly, in such situation $\mathbf{P}_1 = \mathbf{P}_3$ establishing characteristic (h) of the theorem. Other combinations leading to this characteristic consist of: (2.5) with a version of (2.33) having $c_2 + c_3 = 0$; (2.6) with a version of (2.34) having $c_1 + c_2 + c_3 = 1$; three cases in which (2.8) is combined with (2.23) under $c_2 + c_3 = 0$, (2.33), or a version of (2.34) under $c_2 + c_3 = 0$.

Combining (2.5) with (2.31) having $c_1 + c_3 = 1$, $c_2 = -1$ leads, additionally, to $\mathbf{S} = \mathbf{I}_r$, $\mathbf{Y} = \mathbf{0}$, $\mathbf{T} = \mathbf{0}$. These matrix conditions are equivalent to conjunction $\mathbf{P}_1 = \mathbf{P}_3$, $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$ and hence we obtain characteristic (i). The same result follows by combining (2.8) with (2.34) under $c_2 + c_3 = 1$.

Four characteristics left to be established. The first one follows by combining (2.8) with (2.23) under $c_2 + c_3 = 1$. Then, additionally, $c_1 = 1$, $\mathbf{X} = \mathbf{0}$, $\mathbf{S} = \mathbf{0}$, $\mathbf{Y} = \mathbf{T}$, and the triple of matrix conditions can be equivalently expressed as $\mathbf{P}_2 = \mathbf{P}_3, \mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$. Hence, we arrive at characteristic (j) of the theorem.

Next, combining (2.5) with (2.28) leads to $c_1 + c_2 \in \{0, 1\}$, $c_1 + c_3 \in \{0, 1\}$, $\mathbf{X} + \mathbf{S} = \mathbf{I}_r$, $\mathbf{Y} = \mathbf{0}$, $\mathbf{T} = \mathbf{0}$. Utilizing projectors $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ given in (2.3) and (2.4) it is seen that these conditions entail $\mathbf{P}_1 = \mathbf{P}_2 + \mathbf{P}_3, \mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$. In consequence, characteristic (k) is shown. Similarly, this characteristic is attainable by combining (2.8) with (2.25).

Further, combining (2.9) with (2.23) entails $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $\mathbf{X} = \mathbf{0}$, $\mathbf{S} = \mathbf{0}$, $\mathbf{YT} = \mathbf{0}$. Clearly, these matrix equalities hold if and only if $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{0}$, and $\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$. Hence, characteristic (l) is established with the use of indexes "i", "j", and "k".

Finally, combining (2.5) with (2.34) gives $c_1 + c_2 + c_3 \in \{0, 1\}, \mathbf{X} = \mathbf{I}_r, \mathbf{S} = \mathbf{I}_r, \mathbf{Y} = \mathbf{0}, \mathbf{T} = \mathbf{0}$ or, in other words, $\mathbf{P}_1 = \mathbf{P}_2 = \mathbf{P}_3$, leading to characteristic (m). The proof is complete. □

Theorem 1 is supplemented by an analysis showing that the extent in which it generalizes part (a) of Theorem 1 in [2] is indeed essential. The following list indicates that only seven out of 13 characteristics listed in Theorem 1 above have their counterparts in Theorem 1 in [2]. Moreover, two from among the seven characteristics provide generalizations of their counterparts.

Theorem 1	Theorem 1 in [2]
Characteristic (a) generalizes:	characteristics (a_2), (a_3), (a_5), and first cases in characteristics (a_6), (a_7)
Characteristic (b) corresponds to	second cases in characteristics (a_6), (a_7)
Characteristic (d) corresponds to	characteristic (a_4)
Characteristic (h) generalizes	first cases in characteristics (a_8), (a_9)
Characteristic (j) corresponds to	second cases in characteristics (a_8), (a_9)
Characteristic (k) corresponds to	characteristic (a_{10})
Characteristic (l) corresponds to	characteristic (a_1)

The next two theorems provide characteristics of situations in which matrices $\mathbf{P}_i$, $i = 1, 2, 3$, are such that, in addition to (1.3), they satisfy conditions $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3\mathbf{P}_1, \mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2$ and $\mathbf{P}_1\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_1, \mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2$, respectively.

A theorem below generalizes parts (b) and (c) of Theorem 1 in [2].

Theorem 2. *Let $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{P}}$ be nonzero and such that*

$$\mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2, \quad \mathbf{P}_1\mathbf{P}_i = \mathbf{P}_i\mathbf{P}_1, \ i = 2, 3. \tag{2.39}$$

Moreover, let $\mathbf{P}$ be a linear combination of the form (1.1), with nonzero $c_1, c_2, c_3 \in \mathbb{C}$ constituting $\alpha = c_1(c_1 - 1)/c_2c_3$. Then the following list comprises characteristics of all cases in which $\mathbf{P}$ is an idempotent matrix:

(a) $\mathbf{P}_1\mathbf{P}_j = \mathbf{P}_1$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1 - \mathbf{P}_1\mathbf{P}_k$, *hold along with* $c_1 = -1$, $c_j = 2$, $c_k = -1$,
(b) $\frac{1}{4}\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2 = 2\mathbf{P}_k + \mathbf{P}_1\mathbf{P}_j - \mathbf{P}_1\mathbf{P}_k$ *holds along with* $c_1 = \frac{1}{2}$, $c_j = 1$, $c_k = -1$,
(c) $\mathbf{P}_1\mathbf{P}_j = \mathbf{0}$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1\mathbf{P}_k$, *hold along with* $c_1 = 1$, $c_j = 2$, $c_k = -1$,
(d) $\mathbf{P}_1\mathbf{P}_j = \mathbf{0}$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1$, $c_2 + c_3 = 1$, *hold along with* $c_1 + c_k = 0$ *or* $c_1 + c_k = 1$,
(e) $\mathbf{P}_1\mathbf{P}_j = \mathbf{P}_j$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \alpha\mathbf{P}_1 + \mathbf{P}_k - \mathbf{P}_1\mathbf{P}_k$, *hold along with* $2c_1 + c_j = 0$, $c_k = 1$,
(f) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \alpha\mathbf{P}_1$, *hold along with* $2c_1 + c_2 + c_3 = 1$,
(g) $\mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1$, $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{2}$, *hold along with* $c_1 = -\frac{1}{2}$ *or* $c_1 = \frac{1}{2}$,
(h) $\frac{3}{4}\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3$ *holds along with* $c_1 = -\frac{1}{2}$, $c_2 = 1$, $c_3 = 1$,
(i) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{0}$, *hold along with* $c_1 = -1$, $c_2 + c_3 = 1$,
(j) $\mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2 - \mathbf{P}_3$, $4c_2^2(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1$, *hold along with* $c_1 = \frac{1}{2}$, $c_2 + c_3 = 0$,
(k) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{0}$, *hold along with* $c_1 = 1$, $c_2 + c_3 = 1$,

where in characteristics (a)–(e) $j \neq k$, $j, k = 2, 3$.

Proof. Under assumptions (2.39), Eq. (2.1) reduces to

$$\begin{aligned} &c_1(c_1 - 1)\mathbf{P}_1 + c_2(c_2 - 1)\mathbf{P}_2 + c_3(c_3 - 1)\mathbf{P}_3 + 2c_1c_2\mathbf{P}_1\mathbf{P}_2 \\ &\quad + 2c_1c_3\mathbf{P}_1\mathbf{P}_3 + c_2c_3(\mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2) = \mathbf{0}. \end{aligned} \tag{2.40}$$

On account of an obvious relationship $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_2 + \mathbf{P}_3 - \mathbf{P}_2\mathbf{P}_3 - \mathbf{P}_3\mathbf{P}_2$, condition (2.40) can be rewritten in the form

$$\begin{aligned} &c_1(c_1 - 1)\mathbf{P}_1 + c_2(c_2 + c_3 - 1)\mathbf{P}_2 + c_3(c_2 + c_3 - 1)\mathbf{P}_3 \\ &\quad + 2c_1c_2\mathbf{P}_1\mathbf{P}_2 + 2c_1c_3\mathbf{P}_1\mathbf{P}_3 = c_2c_3(\mathbf{P}_2 - \mathbf{P}_3)^2. \end{aligned} \tag{2.41}$$

Sufficiency of the conditions revealed in 11 characteristics provided in the theorem follows by direct verification of criterion (2.41). For the proof of necessity, first observe that (2.41) as well as (2.39) are invariant with respect to interchanging indexes “2” and “3”. Thus, similarly as in the proof of Theorem 1, it is reasonable to introduce two indexes, “j” and “k” say, such that $j, k \in \{2, 3\}$, and to use them, under the assumption that $j \neq k$, to express necessary conditions in a possibly compact way.

In the sequel we utilize projectors $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ of the forms (2.3) and (2.4). In view of Theorem 4.1 in [6], from the idempotency of $\mathbf{X} \in \mathbb{C}_{r,r}$ occurring in (2.4) it follows that there exists a nonsingular matrix $\mathbf{U} \in \mathbb{C}_{r,r}$ such that

$$\mathbf{X} = \mathbf{U}(\mathbf{I}_x \oplus \mathbf{0})\mathbf{U}^{-1}, \tag{2.42}$$

where $x = r(\mathbf{X})$. Clearly, $0 \leqslant x \leqslant r$, and if $x = 0$ then the former, whereas if $x = r$ then the latter, of the summands in representation (2.42) vanishes. A useful observation concerning matrices given in (2.3) and (2.4) is that the first condition in (2.39) ensures that either $\mathbf{XS} \neq \mathbf{SX}$ or $\mathbf{YT} \neq \mathbf{TY}$.

Assume first that $\mathbf{XS} \neq \mathbf{SX}$, which means that $\mathbf{X}$ and $\mathbf{S}$ are nonzero and singular. Utilizing matrix $\mathbf{U}$ used in (2.42), we represent $\mathbf{S}$ as

$$\mathbf{S} = \mathbf{U}\begin{pmatrix}\mathbf{S}_1 & \mathbf{S}_2\\ \mathbf{S}_3 & \mathbf{S}_4\end{pmatrix}\mathbf{U}^{-1},$$

where $\mathbf{S}_1 \in \mathbb{C}_{x,x}$, $\mathbf{S}_2 \in \mathbb{C}_{x,r-x}$, $\mathbf{S}_3 \in \mathbb{C}_{r-x,x}$, and $\mathbf{S}_4 \in \mathbb{C}_{r-x,r-x}$. From $\mathbf{XS} \neq \mathbf{SX}$ it follows that either $\mathbf{S}_2 \neq \mathbf{0}$ or $\mathbf{S}_3 \neq \mathbf{0}$. Furthermore, premultiplying (2.12) by $\mathbf{U}^{-1}$ and postmultiplying it by $\mathbf{U}$ leads to

$$\begin{aligned} &c_1(c_1-1)\begin{pmatrix}\mathbf{I}_x & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{r-x}\end{pmatrix} + c_2(2c_1+c_2-1)\begin{pmatrix}\mathbf{I}_x & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\\ &\quad + c_3(2c_1+c_3-1)\begin{pmatrix}\mathbf{S}_1 & \mathbf{S}_2\\ \mathbf{S}_3 & \mathbf{S}_4\end{pmatrix} + c_2c_3\begin{pmatrix}2\mathbf{S}_1 & \mathbf{S}_2\\ \mathbf{S}_3 & \mathbf{0}\end{pmatrix} = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}. \end{aligned} \tag{2.43}$$

It follows from the equation attributed to the upper-right submatrix in (2.43) when $\mathbf{S}_2$ is nonzero, and lower-left submatrix when $\mathbf{S}_3$ is nonzero, that

$$2c_1 + c_2 + c_3 = 1. \tag{2.44}$$

Substituting (2.44) to (2.12) yields

$$\alpha\mathbf{I}_r = (\mathbf{X} - \mathbf{S})^2, \tag{2.45}$$

where $\alpha = c_1(c_1-1)/c_2c_3$. Clearly, if $\mathbf{P}_1$ is nonsingular, i.e., $\mathbf{P}_1 = \mathbf{I}_n$ or, equivalently, $r = n$, then $\mathbf{P}_2 = \mathbf{WXW}^{-1}$ and $\mathbf{P}_3 = \mathbf{WSW}^{-1}$. Hence, by premultiplying and postmultiplying (2.45) by $\mathbf{W}$ and $\mathbf{W}^{-1}$, respectively, we obtain a stronger version of characteristic (f) of the theorem.

On the other hand, if $\mathbf{P}_1$ is singular, then the latter of the summands in representation (2.3) is present and two situations can occur, namely $\mathbf{YT} \neq \mathbf{TY}$ and $\mathbf{YT} = \mathbf{TY}$. In the former of them, from Theorem in [1] it follows that the idempotency of a linear combination $c_2\mathbf{Y} + c_3\mathbf{T}$ entails $c_2 + c_3 = 1$. Substituting this condition to (2.44) yields $c_1 = 0$, which is in a contradiction with the assumptions. If $\mathbf{YT} = \mathbf{TY}$, then $c_2\mathbf{Y} + c_3\mathbf{T}$ can be idempotent in seven situations characterized by sets (2.5)–(2.11). As already mentioned in the proof of Theorem 1, sets (2.6), (2.7) and (2.10), (2.11) are counterparts of each other obtained by interchanging simultaneously matrices $\mathbf{Y}$ and $\mathbf{T}$ as well as scalars c_2 and c_3, and thus the latter sets in these pairs need not be considered separately.

If both $\mathbf{Y}$ and $\mathbf{T}$ are equal to zero matrices, then from (2.3) and (2.4) it follows that $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$ and $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$. Substituting these conditions along with (2.44) to (2.41) leads to $\alpha\mathbf{P}_1 = (\mathbf{P}_2 - \mathbf{P}_3)^2$. In consequence, characteristic (f) is obtained.

Substituting $c_3 = 1$, i.e., the last condition in set (2.6), to (2.44) leads to $2c_1 + c_2 = 0$. Furthermore, on account of $\mathbf{Y} = \mathbf{0}$, it follows from (2.3) and (2.4) that $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$. Hence, Eq. (2.41) reduces to

$$(\mathbf{P}_2 - \mathbf{P}_3)^2 = \alpha\mathbf{P}_1 + \mathbf{P}_3 - \mathbf{P}_1\mathbf{P}_3,$$

and, in consequence, characteristic (e) is obtained.

From (2.44) it follows that $c_2 + c_3 \neq 1$, for otherwise, as already mentioned, $c_1 = 0$. Thus, the latter condition in set (2.8) reduces to $c_2 + c_3 = 0$ and substituting it to (2.44) leads to $c_1 = \frac{1}{2}$. Another observation is that with $\mathbf{Y} = \mathbf{T}$, (2.3) and (2.4) entail $\mathbf{P}_1(\mathbf{P}_2 - \mathbf{P}_3) = \mathbf{P}_2 - \mathbf{P}_3$. Substituting this condition along with $c_1 = \frac{1}{2}$ and $c_2 + c_3 = 0$ to (2.41) gives $\mathbf{P}_1 = 4c_2^2(\mathbf{P}_2 - \mathbf{P}_3)^2$ and thus characteristic (j) of the theorem is established.

Two sets remain to be considered, namely (2.9) and (2.10). Combining $c_2 = 1$, $c_3 = 1$ with (2.44) gives $c_1 = -\frac{1}{2}$ and substituting these values of scalars c_i, $i = 1, 2, 3$, to (2.41) entails

$$\tfrac{3}{4}\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2 = \mathbf{P}_1(\mathbf{P}_2 + \mathbf{P}_3).$$

Consequently, we arrive at characteristic (h).

Similarly, combining $c_2 = 1$, $c_3 = -1$ with (2.44) gives $c_1 = \frac{1}{2}$ and hence (2.41) reduces to

$$\tfrac{1}{4}\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2 = 2\mathbf{P}_3 + \mathbf{P}_1(\mathbf{P}_2 - \mathbf{P}_3),$$

leading to characteristic (b).

Let us now consider situation in which $\mathbf{XS} = \mathbf{SX}$. Then, in view of $\mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2$, the latter of the summands in representation (2.3) is present and $\mathbf{YT} \neq \mathbf{TY}$. Hence, since a linear combination $c_2\mathbf{Y} + c_3\mathbf{T}$ is idempotent, from Theorem in [1] it follows that $(\mathbf{Y} - \mathbf{T})^2 = \mathbf{0}$ and

$$c_2 + c_3 = 1. \tag{2.46}$$

The remaining part of the proof will be based on an observation that $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$ is a linear combination of three mutually commuting idempotent matrices $\mathbf{I}_r, \mathbf{X}, \mathbf{S}$ and thus – assuming that matrices $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ in a linear combination (1.1) are represented by $\mathbf{I}_r$, $\mathbf{X}$, and $\mathbf{S}$, respectively – we can utilize Theorem 1 of the present paper to characterize its idempotency.

First notice, that characteristics (a), (d), (e), (g), (h), and (l) of Theorem 1 cannot be reconciled with (2.46).

Combining conditions in characteristic (b) of Theorem 1 with (2.46) it follows that either $i = 2$, $j = 3$ or $i = 3$, $j = 2$ holds along with $k = 1$. In the former of these cases, the second matrix condition in (b) leads to $\mathbf{S} = \mathbf{0}$, whereas in the latter case to $\mathbf{X} = \mathbf{0}$. Each of these conditions is in a contradiction with the assumptions of Theorem 1.

Next, we consider characteristic (c) of Theorem 1. Combining first $c_i = -\frac{1}{2}$, $c_j = \frac{1}{2}$, $c_k = \frac{1}{2}$ with (2.46) leads to a conclusion that, in addition to $i = 1$, either $j = 2, k = 3$ or $j = 3, k = 2$. In each of these cases, matrix condition in (c) yields $(\mathbf{X} - \mathbf{S})^2 = \mathbf{I}_r$. Hence, in view of $(\mathbf{Y} - \mathbf{T})^2 = \mathbf{0}$, it follows from (2.3) and (2.4) that

$$(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1. \tag{2.47}$$

This equation holds also when $c_i = -\frac{1}{2}$ is replaced by $c_i = \frac{1}{2}$ (which is an alternate case in (c)), for in this situation matrix condition in characteristic (c) entails $(\mathbf{X} - \mathbf{S})^2 = \mathbf{I}_r$ when $i = 1$ and $\mathbf{X} + \mathbf{S} = \mathbf{I}_r$ when $i = 2$ or $i = 3$. Since these conditions are equivalent (each of them implies $\mathbf{XS} = \mathbf{0}$), this step is concluded by an observation that substituting (2.47) along with possible triplets c_i, $i = 1, 2, 3$, to (2.41) gives $\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3$. Thus, characteristic (g) of the theorem is established.

Characteristic (f) of Theorem 1 is to be considered next. Combining $c_i = 2, c_j = -1, c_k = -1$ with (2.46) shows that four situations are possible, namely: (i) $i = 2$, $j = 3$, $k = 1$, (ii) $i = 2$, $j = 1, k = 3$, (iii) $i = 3$, $j = 2$, $k = 1$, and (iv) $i = 3$, $j = 1$, $k = 2$. In the first of them, $c_1 = -1$, $c_2 = 2$, $c_3 = -1$, and matrix condition in (f) gives $\mathbf{X} = \mathbf{I}_r$. In consequence, (2.3) and (2.4) entail $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$ and substituting this condition along with the values of c_1, c_2, c_3 to (2.41) leads to $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1 - \mathbf{P}_1\mathbf{P}_3$. The considerations concerning the remaining three situations are limited to two observations. The first of them is that characteristic (f) of Theorem 1 is invariant with respect to an interchange of indexes "j" and "k" and thus cases (i) and (ii) as well as (iii) and (iv) correspond to the same situations. The second observation is that cases (i) and (iii) are counterparts of each other obtained by interchanging "2" and "3", and thus conditions obtained above can be expanded to characteristic (a).

The next characteristic of Theorem 1 to be considered is (i). Comparison of conditions on scalars c_1, c_2, c_3 provided therein with (2.46) shows that six cases are to be analyzed, namely: (i) $i = 1, j = 2, k = 3$, (ii) $i = 1, j = 3, k = 2$, (iii) $i = 2, j = 1, k = 3$, (iv) $i = 2, j = 3, k = 1$, (v) $i = 3, j = 1, k = 2$, (vi) $i = 3, j = 2, k = 1$. However, cases (i), (vi) as well as (ii), (iv) and (iii), (v) lead to the same situations. Moreover, cases (i) and (ii) are counterparts of each other obtained by interchanging "2" and "3". Consequently, it suffices to consider two cases only, say, (i) in which $c_1 = -1, c_2 = -1, c_3 = 2$ and (iii) in which $c_2 + c_3 = 1, c_1 = -1$. In the former of them, the first matrix condition in (i) leads to $\mathbf{S} = \mathbf{I}_r$. From the discussion above, it follows that this case is already covered by characteristic (a) of the theorem. On the other hand, in case (iii), by utilizing matrix conditions in characteristic (i) of Theorem 1, we arrive at $\mathbf{X} = \mathbf{I}_r, \mathbf{S} = \mathbf{I}_r$. Hence, from (2.3) and (2.4) it follows that $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1, \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1$ and substituting these conditions along with $c_2 + c_3 = 1, c_1 = -1$ to (2.41) gives $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{0}$, establishing characteristic (i) of the theorem.

Similarly as with respect to characteristic (i) of Theorem 1, also in the case of its characteristic (j) six situations are to be considered. However, as easy to verify by comparing corresponding conditions on scalars c_1, c_2, c_3 with (2.46), in two of them it follows that $c_2 = 0$ and in other two that $c_3 = 0$. Furthermore, in the remaining two situations, from matrix conditions in (j) we obtain $\mathbf{X} = \mathbf{0}, \mathbf{S} = \mathbf{0}$. Thus, each of these six situations is irreconcilable with the assumptions of Theorem 1.

The last two characteristics of Theorem 1 are to be considered. From the second matrix condition in (k) it is seen that $i = 1$, for otherwise $\mathbf{X} = \mathbf{0}$ or $\mathbf{S} = \mathbf{0}$. Consequently, there are four situations to be analyzed, namely when in addition to (2.46) the following conditions are fulfilled: (i) $c_1 + c_2 = 0, c_1 + c_3 = 0$, (ii) $c_1 + c_2 = 0, c_1 + c_3 = 1$, (iii) $c_1 + c_2 = 1, c_1 + c_3 = 0$, (iv) $c_1 + c_2 = 1, c_1 + c_3 = 1$. However, each of situations (ii) and (iii) entails $c_1 = 0$, which contradicts the assumptions, so they are excluded from further considerations. In the remaining two situations, i.e., (i) and (iv), $c_1 = -\frac{1}{2}$ and $c_1 = \frac{1}{2}$, respectively, hold along with $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{2}$. Since matrix conditions corresponding to those situations are equivalent to $(\mathbf{X} - \mathbf{S})^2 = \mathbf{I}_r$, utilizing the same arguments as those used in the proof corresponding to characteristic (c) of Theorem 1, we arrive at the characteristic already listed in the theorem as (g).

Finally, we consider characteristic (m) of Theorem 1. Comparing the last condition given therein with (2.46) shows that $c_1 = -1$. Furthermore, matrix condition in (m) entails $\mathbf{X} = \mathbf{I}_r$, $\mathbf{S} = \mathbf{I}_r$. Hence, from (2.3) and (2.4) it follows that $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1, \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1$, and substituting these conditions to (2.41) gives $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{0}$. Thus, we again arrive at the characteristic already listed in the theorem, this time as (i).

In the last step of the proof, we consider three particular cases not covered by Theorem 1, in which either $\mathbf{X} = \mathbf{0}$ or $\mathbf{S} = \mathbf{0}$. Let us first assume that both these conditions are satisfied. Hence, from (2.3) and (2.4) it follows that $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}, \mathbf{P}_1\mathbf{P}_3 = \mathbf{0}$. Furthermore, the idempotency of a linear combination $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$ (now equal to $c_1\mathbf{I}_r$) entails $c_1 = 1$ and substituting $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}, \mathbf{P}_1\mathbf{P}_3 = \mathbf{0}, c_1 = 1, c_2 + c_3 = 1$ to (2.41) gives $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{0}$. Since this condition combined with either $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$ or $\mathbf{P}_1\mathbf{P}_3 = \mathbf{0}$ implies the other of these relationships, we obtain characteristic (k) of the theorem. On the other hand, if $\mathbf{X} = \mathbf{0}$ and $\mathbf{S} \neq \mathbf{0}$, then the idempotency of $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$ means that $c_1\mathbf{I}_r + c_3\mathbf{S}$ is idempotent. We will separately consider the cases corresponding to $\mathbf{S} = \mathbf{I}_r$ and $\mathbf{S} \neq \mathbf{I}_r$. In the former of them, clearly, either $c_1 + c_3 = 0$ or $c_1 + c_3 = 1$, and, in view of $\mathbf{X} = \mathbf{0}, \mathbf{S} = \mathbf{I}_r$, (2.3) and (2.4) give $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}, \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1$. Hence, on account of (2.46), Eq. (2.41) entails $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1$, and, since combining this condition with either $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$ or $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1$ implies the other of these relationships, characteristic (d) follows. If now

$\mathbf{S} \neq \mathbf{I}_r$, then, in view of (2.46), from Theorem in [1] it follows that $c_1 = 1$, $c_2 = 2$, $c_3 = -1$. Moreover, with $\mathbf{X} = \mathbf{0}$, (2.3) and the left identity in (2.4) yield $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$. With these conditions taken into account, (2.41) entails $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1\mathbf{P}_3$, leading to characteristic (c). We conclude this step with an observation that the case corresponding to $\mathbf{X} \neq \mathbf{0}$ and $\mathbf{S} = \mathbf{0}$ is a counterpart of the previous one obtained by interchanging in the resultant conditions indexes "2" and "3" and was taken into account by introducing indexes "j" and "k" in characteristics (d) and (c) of the theorem. The proof is complete. □

It is noteworthy that only three out of 11 characteristics listed in Theorem 2 have their counterparts in Theorem 1 in [2]. Namely, its characteristic (c) corresponds to characteristics (b_1), (c_1), its characteristic (d) to characteristics (b_3) and (c_3), and its characteristic (k) to characteristics (b_2) and (c_2). Clearly, the remaining eight characteristics in Theorem 2 originate from replacing the assumption (1.2) by (1.3).

The following result corresponds to the situation in which from among three possible pairs of matrices $\mathbf{P}_i$, $i = 1, 2, 3$, occurring in a linear combination (1.1), only $\mathbf{P}_1$ and $\mathbf{P}_2$ commute. It generalizes part (d) of Theorem 1 in [2].

Theorem 3. *Let* $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{P}}$ *be nonzero and such that*

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1, \quad \mathbf{P}_i\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_i,\ i = 1, 2. \tag{2.48}$$

Moreover, let $\mathbf{P}$ *be a linear combination of the form* (1.1), *with nonzero* $c_1, c_2, c_3 \in \mathbb{C}$. *Then the following list comprises characteristics of all cases in which* $\mathbf{P}$ *is an idempotent matrix*:

(a) $(2c_j/c_3)(\mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_j) = \mathbf{P}_3 - (\mathbf{P}_j - \mathbf{P}_3)^2 + (\mathbf{P}_k - \mathbf{P}_3)^2$ *holds along with* $c_1 + c_2 = 0$, $c_k + c_3 = 1$,

(b) $(2c_1/c_3)(\mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_2) = \mathbf{P}_3 + (\mathbf{P}_1 - \mathbf{P}_3)^2 + (\mathbf{P}_2 - \mathbf{P}_3)^2$ *holds along with* $c_1 = c_2$, $3c_1 + c_3 = 1$,

(c) $(2c_1/c_3)\mathbf{P}_1\mathbf{P}_2 = (\mathbf{P}_1 - \mathbf{P}_3)^2 + (\mathbf{P}_2 - \mathbf{P}_3)^2 - \mathbf{P}_3$ *holds along with* $c_1 = c_2$, $c_1 + c_3 = 1$,

(d) $c_1c_2(\mathbf{P}_1 - \mathbf{P}_2)^2 + c_1c_3(\mathbf{P}_1 - \mathbf{P}_3)^2 + c_2c_3(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{0}$ *holds along with* $c_1 + c_2 + c_3 = 1$,

where in characteristic (a) $j \neq k,\ j, k = 1, 2$.

Proof. Under assumptions (2.48), Eq. (2.1) reduces to

$$\begin{aligned} &c_1(c_1 - 1)\mathbf{P}_1 + c_2(c_2 - 1)\mathbf{P}_2 + c_3(c_3 - 1)\mathbf{P}_3 + 2c_1c_2\mathbf{P}_1\mathbf{P}_2 \\ &\quad + c_1c_3(\mathbf{P}_1\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_1) + c_2c_3(\mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2) = \mathbf{0}. \end{aligned} \tag{2.49}$$

Sufficiency of the conditions revealed in four characteristics provided in the theorem follows by direct verification of criterion (2.49). For the proof of necessity, observe that assumptions (2.48) and Eq. (2.49) are invariant with respect to an interchange of indexes "1" and "2". Thus, also in the present proof it is reasonable to use indexes $j, k \in \{1, 2\}$, $j \neq k$, in order to shorten the derivations (and the resultant list) of necessary conditions.

Let $\mathbf{P}_1$ have a representation (2.3) and note that, since $\mathbf{P}_1$ and $\mathbf{P}_2$ commute, the representation of $\mathbf{P}_2$ given in (2.4) can also be utilized in the present proof. From $\mathbf{P}_1\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_1$ it follows, on the one hand, that $\mathbf{P}_1$ is singular (i.e., $0 < r < n$), and, in consequence, the latter of the summands in (2.3) is present, and, on the other hand, that representation of $\mathbf{P}_3$ in (2.4) does not hold. Thus, we represent $\mathbf{P}_3$ as

$$\mathbf{P}_3 = \mathbf{W}\begin{pmatrix}\mathbf{K} & \mathbf{L}\\ \mathbf{M} & \mathbf{N}\end{pmatrix}\mathbf{W}^{-1},$$

with $\mathbf{K} \in \mathbb{C}_{r,r}$, $\mathbf{L} \in \mathbb{C}_{r,n-r}$, $\mathbf{M} \in \mathbb{C}_{n-r,r}$, and $\mathbf{N} \in \mathbb{C}_{n-r,n-r}$, where, on account of $\mathbf{P}_1\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_1$, it is seen that either $\mathbf{L} \neq \mathbf{0}$ or $\mathbf{M} \neq \mathbf{0}$. Consequently, premultiplying and postmultiplying (2.49) by $\mathbf{W}^{-1}$ and $\mathbf{W}$, respectively, yields

$$c_1(c_1-1)\begin{pmatrix}\mathbf{I}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix} + c_2(c_2-1)\begin{pmatrix}\mathbf{X} & \mathbf{0}\\ \mathbf{0} & \mathbf{Y}\end{pmatrix} + c_3(c_3-1)\begin{pmatrix}\mathbf{K} & \mathbf{L}\\ \mathbf{M} & \mathbf{N}\end{pmatrix} + 2c_1c_2\begin{pmatrix}\mathbf{X} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$$
$$+c_1c_3\begin{pmatrix}2\mathbf{K} & \mathbf{L}\\ \mathbf{M} & \mathbf{0}\end{pmatrix} + c_2c_3\begin{pmatrix}\mathbf{XK}+\mathbf{KX} & \mathbf{XL}+\mathbf{LY}\\ \mathbf{YM}+\mathbf{MX} & \mathbf{YN}+\mathbf{NY}\end{pmatrix} = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}. \quad (2.50)$$

Assume first that $\mathbf{L}$ is nonzero and observe that from the equation attributed to the upper-right submatrix in (2.50) it follows that

$$(c_1+c_3-1)\mathbf{L} + c_2(\mathbf{XL}+\mathbf{LY}) = \mathbf{0}. \quad (2.51)$$

Recall that matrices $\mathbf{X}$ and $\mathbf{Y}$ are idempotent (parenthetically notice that since $\mathbf{P}_1$ is nonzero and singular, both of them are necessarily present in (2.4)), and thus there exist nonsingular matrices $\mathbf{U} \in \mathbb{C}_{r,r}$ and $\mathbf{V} \in \mathbb{C}_{n-r,n-r}$ such that

$$\mathbf{X} = \mathbf{U}(\mathbf{I}_x \oplus \mathbf{0})\mathbf{U}^{-1} \quad \text{and} \quad \mathbf{Y} = \mathbf{V}(\mathbf{I}_y \oplus \mathbf{0})\mathbf{V}^{-1}, \quad (2.52)$$

where $x = r(\mathbf{X})$ and $y = r(\mathbf{Y})$. Clearly, $0 \leqslant x \leqslant r$ and $0 \leqslant y \leqslant n-r$, and if $x = 0$ and/or $y = 0$ then the former, whereas if $x = r$ and/or $y = n-r$ then the latter, of the summands in the left and/or right identities in (2.52) vanish. Denoting matrices $\mathbf{I}_x \oplus \mathbf{0}$ and $\mathbf{I}_y \oplus \mathbf{0}$ occurring in (2.52) by $\mathbf{D}_\mathbf{X}$ and $\mathbf{D}_\mathbf{Y}$, respectively, gives

$$\mathbf{XL}+\mathbf{LY} = \mathbf{U}(\mathbf{D}_\mathbf{X}\mathbf{U}^{-1}\mathbf{LV} + \mathbf{U}^{-1}\mathbf{LVD}_\mathbf{Y})\mathbf{V}^{-1},$$

and, consequently, from (2.51) we obtain

$$(c_1+c_3-1)\mathbf{U}^{-1}\mathbf{LV} + c_2(\mathbf{D}_\mathbf{X}\mathbf{U}^{-1}\mathbf{LV} + \mathbf{U}^{-1}\mathbf{LVD}_\mathbf{Y}) = \mathbf{0}. \quad (2.53)$$

Notice that from $\mathbf{L} \neq \mathbf{0}$ and the nonsingularity of $\mathbf{U}$ and $\mathbf{V}$ it follows that $\mathbf{U}^{-1}\mathbf{LV} \neq \mathbf{0}$. Let us represent matrix $\mathbf{L}$ as

$$\mathbf{L} = \mathbf{U}\begin{pmatrix}\mathbf{L}_1 & \mathbf{L}_2\\ \mathbf{L}_3 & \mathbf{L}_4\end{pmatrix}\mathbf{V}^{-1}, \quad (2.54)$$

with $\mathbf{L}_1 \in \mathbb{C}_{x,y}$, $\mathbf{L}_2 \in \mathbb{C}_{x,n-r-y}$, $\mathbf{L}_3 \in \mathbb{C}_{r-x,y}$, and $\mathbf{L}_4 \in \mathbb{C}_{r-x,n-r-y}$, where: $\mathbf{L}_1$ and $\mathbf{L}_2$ vanish if $x = 0$; $\mathbf{L}_3$ and $\mathbf{L}_4$ vanish if $x = r$; $\mathbf{L}_1$ and $\mathbf{L}_3$ vanish if $y = 0$; and $\mathbf{L}_2$ and $\mathbf{L}_4$ vanish if $y = n-r$. Substituting (2.54) to (2.53) yields

$$(c_1+c_3-1)\begin{pmatrix}\mathbf{L}_1 & \mathbf{L}_2\\ \mathbf{L}_3 & \mathbf{L}_4\end{pmatrix} + c_2\begin{pmatrix}2\mathbf{L}_1 & \mathbf{L}_2\\ \mathbf{L}_3 & \mathbf{0}\end{pmatrix} = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}, \quad (2.55)$$

and taking into account that at least one of $\mathbf{L}_i$s, $i = 1, \ldots, 4$, is nonzero, it follows from (2.55) that one of the conditions

$$\text{(i) } c_1+2c_2+c_3 = 1, \quad \text{(ii) } c_1+c_2+c_3 = 1, \quad \text{(iii) } c_1+c_3 = 1, \quad (2.56)$$

must be satisfied. As easy to observe, if $x = 0$ or $y = 0$, then condition (i) does not hold, whereas if $x = r$ or $y = n - r$, then condition (iii) does not hold. (Parenthetically notice that conditions (2.56) are mutually excluding.) Combining (2.56) with the fact that assumptions (2.48) and Eq. (2.49) do not change upon an interchange of indexes "1" and "2", results in another set of necessary conditions, namely

$$\text{(iv) } 2c_1 + c_2 + c_3 = 1, \quad \text{(v) } c_1 + c_2 + c_3 = 1, \quad \text{(vi) } c_2 + c_3 = 1, \tag{2.57}$$

of which at least one must be satisfied.

To conclude this step of the proof we need to consider nine situations corresponding to the possible conjunctions of one condition from (2.56) and one condition from (2.57). However, two of these conjunctions, i.e., (i), (v) and (iii), (v), lead to $c_2 = 0$, and the other two, i.e., (ii), (iv) and (ii), (vi), lead to $c_1 = 0$. Moreover, situations corresponding to conjunctions (i), (vi) and (iii), (iv) are counterparts of each other obtained by interchanging indexes "1" and "2". Consequently, four situations are to be considered.

Combining (i) with (iv) shows that $c_1 = c_2$, $3c_1 + c_3 = 1$. With these relationships taken into account, (2.49) yields matrix condition in characteristic (b) of the theorem. Next, from conjunction (i) and (vi) we obtain $c_1 + c_2 = 0$, $c_2 + c_3 = 1$. Substituting these conditions into (2.49), and including also the case obtained by interchanging indexes "1" and "2" leads to matrix condition in (a). Further, since conditions characterizing cases (ii) and (v) are the same, we cannot obtain more information about scalars c_1, c_2, c_3 than just $c_1 + c_2 + c_3 = 1$. In this situation, however, Eq. (2.49) can be rewritten as in characteristic (d). Finally, combining (iii) with (vi) entails $c_1 = c_2$, $c_1 + c_3 = 1$ and these conditions are in characteristic (c) of the theorem accompanied by an equality obtained from (2.49).

The proof is concluded with an observation that if $\mathbf{M} \neq \mathbf{0}$, then from the equation attributed to the lower-left submatrix in (2.50), an analogue condition to (2.51) is obtained, simply with $\mathbf{L}$ replaced by $\mathbf{M}$ and, additionally, $\mathbf{X}$ and $\mathbf{Y}$ interchanged. Thus, following the steps of the proof corresponding to the situation in which $\mathbf{L} \neq \mathbf{0}$, also when $\mathbf{M} \neq \mathbf{0}$ we would obtain characteristics already listed in the theorem. The proof is complete. □

In a comment to Theorem 3 we emphasize that the extent in which it generalizes part (d) of Theorem 1 in [2] is essential, for only two out of four characteristics listed therein have their counterparts in Theorem 1 in [2]. Namely, characteristic (c) generalizes case (d_1), whereas characteristic (d) generalizes case (d_2).

3. Additional results

In this section we consider situations in which idempotent matrices $\mathbf{P}_i$, $i = 1, 2, 3$, occurring in a linear combination (1.1), are Hermitian. As already mentioned, such situations are of particular interest from the point of view of possible applications in statistics.

We begin with arguments showing that all 13 characteristics listed in Theorem 1 remain valid when $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{OP}}$. The reasoning is based on the fact that, since $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ are mutually commuting idempotent matrices, they are simultaneously diagonalizable, i.e., there exists a nonsingular matrix $\mathbf{U} \in \mathbb{C}_{n,n}$, say, such that $\mathbf{D}_i = \mathbf{U}\mathbf{P}_i\mathbf{U}^{-1}$, $i = 1, 2, 3$, are diagonal matrices with diagonal entries being equal to either zero or one; see e.g., Theorem 1.3.19 in [3]. Thus, $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3 \in \mathbb{C}_n^{\mathsf{OP}}$. In consequence, it is obvious that if given mutually commuting $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{OP}}$ satisfy any matrix condition (for instance one of those occurring in characteristics (i)–(m) of Theorem 1), then $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3 \in \mathbb{C}_n^{\mathsf{OP}}$ satisfy this condition as well.

The following theorem shows that when $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{OP}}$, then six of 11 characteristics listed in Theorem 2 are no longer valid, and two from among five characteristics which are valid have more restrictive forms.

Theorem 4. *Let* $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{OP}}$ *be nonzero and such that conditions* (2.39) *are satisfied. Moreover, let* $\mathbf{P}$ *be a linear combination of the form* (1.1), *with nonzero* $c_1, c_2, c_3 \in \mathbb{C}$ *constituting* $\alpha = c_1(c_1 - 1)/c_2c_3$. *Then* $\alpha > 0$ *and the following list comprises characteristics of all cases in which* $\mathbf{P}$ *is an idempotent* (*and Hermitian*) *matrix*:

(a) $\frac{1}{4}\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2 = 2\mathbf{P}_k + \mathbf{P}_1\mathbf{P}_j - \mathbf{P}_1\mathbf{P}_k$ *holds along with* $c_1 = \frac{1}{2}$, $c_j = 1$, $c_k = -1$,
(b) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \alpha\mathbf{P}_1$, *hold along with* $2c_1 + c_2 + c_3 = 1$,
(c) $\frac{3}{4}\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3$ *holds along with* $c_1 = -\frac{1}{2}$, $c_2 = 1$, $c_3 = 1$,
(d) $\mathbf{P}_1\mathbf{P}_j = \mathbf{P}_j$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \alpha\mathbf{P}_1 + \mathbf{P}_k - \mathbf{P}_1\mathbf{P}_k$, *hold along with* $2c_1 + c_j = 0$, $c_k = 1$, $c_1 < 1$,
(e) $\mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2 - \mathbf{P}_3$, $4c_2^2(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1$, *hold along with* $c_1 = \frac{1}{2}$, $c_2 + c_3 = 0$, $c_2 \in \mathbb{R}$,

where in characteristics (a) *and* (d) $j \neq k$, $j, k = 2, 3$.

Proof. It is known that for every $\mathbf{K} \in \mathbb{C}_n^{\mathsf{OP}}$, there exists a unitary matrix $\mathbf{U} \in \mathbb{C}_{n,n}$ such that $\mathbf{K} = \mathbf{U}(\mathbf{I}_k \oplus \mathbf{0}_{n-k})\mathbf{U}^*$, where $k = \mathrm{r}(\mathbf{K})$. Thus, since $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{OP}}$, we can assume that matrix $\mathbf{W}$ occurring in (2.3) and (2.4) satisfies $\mathbf{W}^{-1} = \mathbf{W}^*$. In consequence, it is seen from (2.4) that $\mathbf{X}, \mathbf{S} \in \mathbb{C}_r^{\mathsf{OP}}$ and $\mathbf{Y}, \mathbf{T} \in \mathbb{C}_{n-r}^{\mathsf{OP}}$.

Similarly as in the proof of Theorem 2, assume first that $\mathbf{XS} \neq \mathbf{SX}$. Then, from (2.45) it follows that

$$\alpha\mathbf{I}_r = (\mathbf{X} - \mathbf{S})^*(\mathbf{X} - \mathbf{S}). \tag{3.1}$$

Combining (3.1) with a known fact that for every $\mathbf{K} \in \mathbb{C}_{m,n}$, the product $\mathbf{K}^*\mathbf{K}$ is nonnegative definite, and, moreover, $\mathbf{K}^*\mathbf{K} = \mathbf{0}$ if and only if $\mathbf{K} = \mathbf{0}$, entails $\alpha > 0$.

Recall that under the assumption $\mathbf{XS} \neq \mathbf{SX}$, five characteristics of Theorem 2 were obtained, namely (b), (e), (f), (h), and (j). Three of them, i.e., (b), (f), and (h) correspond to characteristics (a), (b), and (c) of the present theorem, respectively. Now, from conditions $2c_1 + c_j = 0$ and $c_k = 1$, being a part of characteristic (e) of Theorem 2, we get $\alpha = (1 - c_1)/2$, regardless of whether $j = 2, k = 3$ or $j = 3, k = 2$. Hence, $c_1 < 1$, leading to characteristic (d) of the theorem. Analogously, from the scalar conditions in characteristic (j) of Theorem 2 we obtain $\alpha = 1/4c_2^2$. Thus, $c_2^2 > 0$, which means that $c_2 \in \mathbb{R}$, and we arrive at characteristic (e) of the theorem.

Consider now situation in which $\mathbf{XS} = \mathbf{SX}$. Recall that from the proof of Theorem 2 it follows that then $\mathbf{YT} \neq \mathbf{TY}$ and $(\mathbf{Y} - \mathbf{T})^2 = \mathbf{0}$. However, from the latter of these conditions we have $(\mathbf{Y} - \mathbf{T})^*(\mathbf{Y} - \mathbf{T}) = \mathbf{0}$, which yields $\mathbf{Y} = \mathbf{T}$, being in a contradiction with the former condition. The proof is complete. □

The last result refers to Theorem 3 and shows that two out of four characteristics listed therein are no longer valid when $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{OP}}$.

Theorem 5. *Let* $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{OP}}$ *be nonzero and such that conditions* (2.48) *are satisfied. Moreover, let* $\mathbf{P}$ *be a linear combination of the form* (1.1), *with nonzero* $c_1, c_2, c_3 \in \mathbb{C}$. *Then the* *following list comprises characteristics of all cases in which* $\mathbf{P}$ *is an idempotent* (*and Hermitian*) *matrix*:

(a) $(2c_1/c_3)(\mathbf{P}_1\mathbf{P}_2-\mathbf{P}_1-\mathbf{P}_2) = \mathbf{P}_3+(\mathbf{P}_1-\mathbf{P}_3)^2+(\mathbf{P}_2-\mathbf{P}_3)^2$ *holds along with* $c_1 = c_2$, $3c_1 + c_3 = 1$,
(b) $c_1c_2(\mathbf{P}_1-\mathbf{P}_2)^2+c_1c_3(\mathbf{P}_1-\mathbf{P}_3)^2+c_2c_3(\mathbf{P}_2-\mathbf{P}_3)^2=\mathbf{0}$ *holds along with* $c_1 + c_2 + c_3 = 1$.

Proof. Referring to representations of $\mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{P}}$ utilized in the proof of Theorem 3, and taking into account that these matrices are now Hermitian, we can represent them as

$$\mathbf{P}_2 = \mathbf{W}\begin{pmatrix}\mathbf{X} & \mathbf{0}\\ \mathbf{0} & \mathbf{Y}\end{pmatrix}\mathbf{W}^* \quad \text{and} \quad \mathbf{P}_3 = \mathbf{W}\begin{pmatrix}\mathbf{K} & \mathbf{L}\\ \mathbf{M} & \mathbf{N}\end{pmatrix}\mathbf{W}^*, \tag{3.2}$$

where $\mathbf{X}, \mathbf{K} \in \mathbb{C}_{r,r}$ and $\mathbf{Y}, \mathbf{N} \in \mathbb{C}_{n-r,n-r}$. From the left identity in (3.2) it is seen that $\mathbf{X} \in \mathbb{C}_r^{\mathsf{OP}}$, $\mathbf{Y} \in \mathbb{C}_{n-r}^{\mathsf{OP}}$, whereas from the right identity we get $\mathbf{K}^* = \mathbf{K}$, $\mathbf{N}^* = \mathbf{N}$, and $\mathbf{L}^* = \mathbf{M}$. Moreover, the idempotency (along with Hermitancy) of $\mathbf{P}_3$ entails

$$\mathbf{N} = \mathbf{L}^*\mathbf{L} + \mathbf{N}^2. \tag{3.3}$$

In the remaining part of the proof we will show that characteristics (a) and (c) of Theorem 3 do not hold when $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ are Hermitian. We will refer to condition

$$c_2(c_2-1)\mathbf{Y} + c_3(c_3-1)\mathbf{N} + c_2c_3(\mathbf{YN}+\mathbf{NY}) = \mathbf{0}, \tag{3.4}$$

obtained from the equation attributed to the lower-right submatrix in (2.50). Substituting $c_1 + c_2 = 0$ and $c_2 + c_3 = 1$, being a part of characteristic (a) of Theorem 3 with $k = 2$, to (3.4) gives

$$\mathbf{N} = \mathbf{YN} + \mathbf{NY} - \mathbf{Y}. \tag{3.5}$$

Combining (3.3) with (3.5) leads to $\mathbf{L}^*\mathbf{L} + \mathbf{N}^2 = \mathbf{YN} + \mathbf{NY} - \mathbf{Y}$. Since $\mathbf{Y}$ and $\mathbf{N}$ are Hermitian, this condition can be equivalently expressed as

$$\mathbf{L}^*\mathbf{L} + (\mathbf{Y}-\mathbf{N})^*(\mathbf{Y}-\mathbf{N}) = \mathbf{0},$$

i.e., as a sum of two nonnegative definite matrices. Hence, $\mathbf{L} = \mathbf{0}$, and from (2.3) and (3.2) it follows that $\mathbf{P}_1$ and $\mathbf{P}_3$ commute, what is in a contradiction with assumptions (2.48). The same contradiction is obtained if in characteristic (a) of Theorem 3, $k = 1$. This fact is seen by noticing that the roles of $\mathbf{P}_1$ and $\mathbf{P}_2$ are symmetrical in both Theorem 3 and the present theorem.

The proof is concluded by an observation that substituting $c_1 = c_2$ and $c_1 + c_3 = 1$, i.e., conditions occurring in characteristic (c) of Theorem 3, to (3.4) leads to (3.5) as well. □

Acknowledgments

The authors are very thankful to a referee for valuable comments on earlier version of the paper, which resulted in rewriting the proof of Theorem 1.

References

[1] J.K. Baksalary, O.M. Baksalary, Idempotency of linear combinations of two idempotent matrices, Linear Algebra Appl. 321 (2000) 3–7.

[2] O.M. Baksalary, Idempotency of linear combinations of three idempotent matrices two of which are disjoint, Linear Algebra Appl. 388 (2004) 67–78.
[3] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, UK, 1985.
[4] H. Özdemir, A.Y. Özban, On idempotency of linear combinations of idempotent matrices, Appl. Math. Comput. 159 (2004) 439–448.
[5] C.R. Rao, S.K. Mitra, Generalized Inverse of Matrices and Its Applications, Wiley, New York, 1971.
[6] F. Zhang, Matrix Theory, Springer, New York, 1999.